ASYMPTOTIC FORMS OF COULOMB WAVE FUNCTIONS, II

by

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9. Introduction

In a preceding report [2] we obtained asymptotic forms of certain solutions of the differential equation

$$(9.1)\ \frac{d^2y}{dx^2} + \nu^2 \left[\frac{x}{1+x} - \frac{L(L+1)}{\nu^2(1+x)^2} \right] y = 0.$$

These asymptotic forms, (4.25) and (4.26), were expressed in terms of Airy functions: they provide asymptotic representations of solutions of (9.1) for $x > -1$ and ν large and positive, and these representations hold uniformly in x provided that $x \geq -1 + \epsilon$ $(\epsilon > 0)$ in the case of (4.25) and $\nu x \geq k$ in the case of (4.26).

When applied to Coulomb wave functions $F_L(\eta, \rho)$ and $G_L(\eta, \rho)$, we obtained the asymptotic forms (6.15). The asymptotic representation of the irregular Coulomb wave function $G_L(\eta, \rho)$ holds for all $\rho > 0$ and large η; that for the regular Coulomb wave function $F_L(\eta, \rho)$ gives a significant result only for $\rho \geq 2\eta$, i.e., $x \geq 0$.

In the present report we fill the gap thus left in that we give asymptotic forms for $F_L(\eta, \rho)$ valid when $0 < \rho \leq 2\eta$. In addition, we also give a simplified asymptotic form for F_L when $\rho \geq 2\eta$, and a simplified form for G_L for all ρ. Unlike (6.15), the simplified forms do not provide asymptotic representations for fixed η as $\rho \to \infty$, and they are less accurate than (6.15) when ρ is very large. They are, however, very useful when ρ and η are of the same order of magnitude.

The asymptotic forms given in [2] were established by a comparison of (1) with the differential equation (2.7). The solutions of these two equations were "matched" at $x = \infty$, and the constants were determined by fixing ν and making $x \to \infty$ (or $\rho \to \infty$). Now, the simplified asymptotic forms to be given in this report do not hold when ν (or η) is fixed and $x \to \infty$ (or $\rho \to \infty$), and a direct proof along the lines followed in the earlier report does not seem to be feasible. In the range covered in the earlier report, i.e., $\rho > 0$ in the case of G_L, and $\rho \geq 2\eta$ in the case of F_L, we establish the validity of the new asymptotic forms by comparison with those obtained in [2]. In the remaining case, that of F_L when $0 < \rho < 2\eta$, we first provide an auxiliary asymptotic form in terms of modified Bessel functions of order $2L + 1$, and then establish the Airy function approximation by comparison with the Bessel function approximation. The latter is of interest in its own right; and it is obtained by a comparison of (1) with (10.1) below, the solutions of these two equations being "matched" at $x = -1$ (i.e. $\rho = 0$).

The notations and conventions of the earlier report will be used without explanation, and equations and sections of that report will be referred to simply by their numbers.

10. Outline of the Bessel function approximation

The need for an asymptotic approximation for the regular Coulomb wave function in the interval to the left of the transition point has been discussed in the previous section. A comparison differential equation will be sought which has the same characteristics as (9.1) to the left of the transition point, namely monotone solutions for $-1 < x \leq 0$ and a single regular singularity at $x = -1$. The simplest equation of this type is the differential equation satisfied by the modified Bessel functions $\xi^{1/2} I_p(\nu \xi^{\gamma})$. Moreover, the choice $p\gamma = L + 1/2$ of the order p of the Bessel function will guarantee that the exponents at -1 are identical with those of equation (9.1).

Accordingly, the differential equation

$$(10.1) \quad U'' + \left[-\nu^2 \chi'^2 - (p^2 - 1/4) \frac{\chi'^2}{\chi^2} + 1/2 \{\chi, x\} \right] U = 0$$

satisfied by the function [6; p. 98 (17), (18)]

$$U_0 = \left(\frac{\chi}{\chi'} \right)^{1/2} I_p(\nu\chi)$$

will be used as a comparison equation for (9.1). Here, the primes denote differentiation with respect to x.

The dominant terms for large ν in (9.1) and (10.1) are made to coincide by the choice

$$(10.2) \quad \chi' = \left(\frac{-x}{1+x} \right)^{1/2}; \quad \chi(x) = \int_{-1}^{x} \left(\frac{-t}{1+t} \right)^{1/2} dt \qquad -1 < x < 0.$$

Here, the square root is positive, and the fixed limit of integration has been determined so that (9.1) and (10.1) have the same singularities. The order p of the Bessel function is a parameter at our disposal, to be determined so that the two differential equations will be as similar as possible near the singularity.

A fundamental set of solutions of (10.1) is

$$(10.3)\quad U_0 = \left(\frac{\chi}{\chi'}\right)^{1/2} I_p(\nu\chi); \qquad U_1 = \left(\frac{\chi}{\chi'}\right)^{1/2} K_p(\nu\chi),$$

where I_p and K_p are the modified Bessel functions of the first and third kind respectively, of order p. Let u_0 be the solution of (9.1) with the same behavior as U_0 near the singularity -1. When these solutions are compared by means of the Volterra integral equation

$$(10.4)\quad u_0(x) = U_0(x) + \int_{-1}^{x} K(x, t)\, F(t)\, u_0(t)\, dt$$

where

$$(10.5)\quad K(x, t) = \Delta^{-1}\,[U_0(t)\, U_1(x) - U_1(t)\, U_0(x)],$$

$$\Delta = U_0(t)\, U_1'(t) - U_1(t)\, U_0'(t) = -1,$$

and

$$(10.6)\quad F(x) = \frac{L(L+1)}{(1+x)^2} - (p^2 - 1/4)\,\frac{\chi'^2}{\chi^2} + 1/2\{\chi, x\}.$$

If $U_0(x)$ is a solution of (10.1), and if $u_0(x)$ is a solution of (10.4) for which the integral on the right can be twice differentiated under the integral sign, then according to a lemma of T.M. Cherry [1], $u_0(x)$ satisfies the differential equation (9.1) and also it shows the same behavior at $x = -1$ as $U_0(x)$. In this regard, $u_0(x)$ will be identified by its asymptotic behavior as $x \to -1$. The integral in (10.4) may be improper provided it converges and may be differentiated twice under the integral sign. In the next section, an analytic solution for (10.4) will be established in the interval* $-1 < x \leq -\epsilon$ with the property that the integral term is $O(1/\nu)$ for large ν. By Cherry's lemma, this will lead to an asymptotic form of the solution of the differential equation (9.1) valid for large ν, $-1 < x \leq -\epsilon$.

* In sections 10-13, ϵ denotes a small positive number.

The function $\chi(x)$ defined by (10.2) is given explicitly by

$$\begin{aligned}(10.7)\quad \chi(x) &= [-x(1+x)]^{½} - ½\cos^{-1}(2x+1) + ½\pi \qquad -1 < x \leq 0\\ &= -\frac{2}{3}[-\phi(x)]^{3/2} + ½\pi,\end{aligned}$$

where $\phi(x)$ is defined as an analytic continuation of the solution (3.3) of (2.9), as explained in sec. 2. In (10.7), $\cos^{-1}\tau$ denotes the principal branch of the inverse cosine, $0 \leq \cos^{-1}\tau \leq \pi$. The following results can be verified

$$(10.8)\qquad \chi(x) = 2(1+x)^{1/2} + O[(1+x)^{3/2}] \qquad \text{as } x \to -1,$$

$$\frac{\chi'(x)^2}{\chi(x)^2} = \frac{-x}{4(1+x)^2} + O\left(\frac{1}{1+x}\right) \qquad \text{as } x \to -1,$$

$$\{\chi(x), x\} = \frac{3}{8x^2(1+x)^2} - \frac{1}{x^2(1+x)} .$$

Hence, (10.6) can be written

$$\begin{aligned}F(x) &= \frac{L(L+1)}{(1+x)^2} - (p^2 - ¼)\left[\frac{-x}{4(1+x)^2} + O\left(\frac{1}{1+x}\right)\right]\\ &\quad + \frac{3}{16x^2(1+x)^2} - \frac{1}{2x^2(1+x)} \qquad x \to -1,\\ &= \frac{1}{(1+x)^2}\,[L(L+1) - (½p)^2 + ¼]\\ &\quad + \frac{1}{(1+x)^2}\left\{\left[\left(\frac{p}{2}\right)^2 - \frac{1}{16}\right](1+x) - \frac{3}{16}\left(1 - \frac{1}{x^2}\right)\right\} + O\left(\frac{1}{1+x}\right)\end{aligned}$$

With the choice

$$(10.9)\qquad p = 2L + 1$$

for the order of the Bessel functions, the first term vanishes. Since the second term is $O[(1+x)^{-1}]$, $F(x)$ is also $O[(1+x)^{-1}]$ as $x \to -1$. Furthermore, $\{\chi, x\}$ is bounded and χ is bounded away from zero for $-1 + \epsilon \leq x \leq -\epsilon$, so that $F(x)$ is $O[(1+x)^{-1}]$ in the entire interval

$-1 < x \leq -\epsilon$. Hence, there exists a constant B such that

$$(10.10)\quad |F(x)| \leq \frac{B}{1+x} \qquad -1 < x \leq -\epsilon.$$

11. Solution of the integral equation for $u_0(x)$

A detailed discussion of the integral equation (10.4) will be given in this section. It will be shown that there exists a unique analytic solution $u_0(x)$ which is asymptotic to $U_0(x)$ for large values of ν.

In terms of the functions

$$(11.1)\quad \psi_0(x) = u_0(x)/U_0(x),$$

the integral equation (10.4) is rewritten

$$(11.2)\quad \psi_0(x) = 1 + \int_{-1}^{x} R_0(x, t)\, F(t)\, \psi_0(t)\, dt$$

where

$$(11.3)\quad R_0(x, t) = U_0(t)\, U_1(t) - U_0^2(t)\, \frac{U_1(x)}{U_0(x)}.$$

For real positive values of z and for $p \geq 0$, that is $L \geq -\frac{1}{2}$, the Bessel function $I_{2L+1}(z)$ has no zeros, so that $U_0(x)$ is different from zero, $-1 < x \leq -\epsilon$. The function $R_0(x, t)$ is then a well-determined combination of solutions of (10.1) for $-1 < t \leq x \leq -\epsilon$.

By the properties of the modified Bessel functions [6], it follows that there exist positive constants m, M and N such that

$$(11.4)\quad m \leq \frac{e^{-z}(1+z)^{p+\frac{1}{2}}\, I_p(z)}{z^p} \leq M \qquad 0 < z < \infty, \quad p \geq 0$$

$$\frac{e^{z} z^{p} K_p(z)}{(1+z)^{p-\frac{1}{2}}} \leq N, \qquad 0 < z < \infty, \quad p > 0.$$

Hence, from (10.3), (11.3) and (11.4),

$$|R_0(x, t)| \leq \left\{ \frac{M^2 N}{m} \frac{[1+\nu\chi(x)]^{4L+2}}{[\nu\chi(x)]^{4L+2}} \frac{[\nu\chi(t)]^{4L+2}}{[1+\nu\chi(t)]^{4L+3}} \right.$$

$$\left. \times \exp\{-2\nu[\chi(x) - \chi(t)]\} + \frac{MN}{1+\nu\chi(t)} \right\} \frac{\chi(t)}{\chi'(t)}$$

From (10.2) it follows that $0 < \chi(t) \leq \chi(x)$ for $-1 < t \leq x \leq -\epsilon$. Hence there exists a constant A_1 such that

$$|R_0(x, t)| \leq \frac{A_1}{\nu\chi'(t)} \qquad \nu > \nu_0$$

or from (10.2),

$$|R_0(x, t)| \leq \frac{A(1+t)^{½}}{\nu} \qquad -1 < t \leq x \leq -\epsilon.$$

From (10.10),

$$(11.5) \quad |R_0(x, t)\, F(t)| \leq \frac{AB}{\nu(1+t)^{½}} \qquad -1 < t \leq x \leq -\epsilon.$$

A solution of the integral equation (11.2) can now be constructed by the method of successive approximations. The series

$$(11.6) \quad \psi_0(x) = \sum_{n=0}^{\infty} \psi_0^{(n)}(x),$$

where

$$(11.7) \quad \psi_0^{(0)}(x) = 1, \qquad \psi_0^{(n)}(x) = \int_{-1}^{x} R_0(x, t) F(t)\, \psi_0^{(n-1)}(t)\, dt \qquad n \geq 1,$$

defines a formal solution of the integral equation. It follows from (11.5) and (11.6) by mathematical induction on the positive integer n that

$$|\psi_0^{(n)}(x)| \leq \left[\frac{2AB(1+x)^{½}}{\nu}\right]^n \frac{1}{n!} \qquad n = 0, 1, 2, \dots$$

The constants A and B are independent of x, ν, and n, and consequently the series (11.5) is absolutely and uniformly convergent to a regular function $\psi_0(x)$, $-1 \leq x < -\epsilon$, $\nu > \nu_0$, which represents the only continuous solution of the integral equation (11.2). Furthermore,

$$|\psi_0(x) - 1| \leq \exp\left[\frac{2AB(1+x)^{½}}{\nu}\right] - 1 \qquad -1 \leq x \leq -\epsilon.$$

A weaker statement is that

$$(11.8) \quad |\psi_0(x) - 1| \leq \frac{2AB(1+x)^{½}/\nu}{1 - [2AB(1+x)^{½}/\nu]} \quad -1 \leq x \leq -\epsilon, \; \frac{\nu}{(1+x)^{½}} > 2AB.$$

Hence,

$$(11.9)\quad \psi_0(x) = 1 + O\left(\frac{1}{\nu}\right) \qquad -1 \le x \le -\epsilon < 0, \quad \nu > \nu_0$$

$$(11.10)\quad \psi_0(x) = 1 + O\left[\frac{(1+x)^{1/2}}{\nu}\right] \qquad \frac{\nu}{(1+x)^{1/2}} > K = 4AB.$$

A solution $\psi_0(x)$ of the integral equation (11.2) which is analytic for $-1 \le x \le -\epsilon$ has now been obtained with the alternate asymptotic forms (11.9) and (11.10). Hence, the function $u_0(x)$ given by (11.1) is a solution of the integral equation (10.4) with the asymptotic behavior

$$(11.11)\quad u_0 = U_0\left[1 + O\left(\frac{1}{\nu}\right)\right] \qquad -1 \le x \le -\epsilon, \quad \nu > \nu_0;$$

$$(11.12)\quad u_0 = U_0\left\{1 + O\left[\frac{(1+x)^{1/2}}{\nu}\right]\right\} \qquad \frac{\nu}{(1+x)^{1/2}} > K.$$

Since the integral on the right of (10.4) can be differentiated twice under the integral sign, it follows from Cherry's lemma that the function $u_0(x)$ represents an analytic solution of the differential equation (9.1) with the asymptotic behavior (11.11) and (11.12).

Asymptotic forms for the derivatives $u_0'(x)$ will now be obtained. From the properties of the modified Bessel functions [6], there exist positive constants m', M', and N' such that

$$(11.13)\quad m' \le \frac{e^{-z}(1+z)^{p-1/2}\, I_p'(z)}{z^{p-1}} \le M' \ ;$$

$$\frac{e^{z}z^{p+1}}{(1+z)^{p+1/2}}\, K_p'(z) \le N'.$$

Let*

$$(11.14)\quad Z_0(x) = I_{2L+1}[\nu\chi(x)] = \left(\frac{\chi'}{\chi}\right)^{1/2} U_0(x);$$

$$Z_1(x) = K_{2L+1}[\nu\chi(x)] = \left(\frac{\chi'}{\chi}\right)^{1/2} U_1(x),$$

* The functions $Z_0(x)$, $Z_1(x)$, and $z_0(x)$ are to be distinguished from similarly designated functions in sec. 5.

and

(11.15) $z_0(x) = \left(\dfrac{\chi}{\chi}\right)^{\frac{1}{2}} u_0(x).$

Then it follows from (10.4) that [see equation (5.3)]

(11.16) $\dfrac{z_0'(x)}{Z_0'(x)} = 1 + \int_{-1}^{x} R_0'(x, t)\, F(t)\, \psi_0(t)\, dt$

where

(11.17) $R_0'(x, t) = U_0^2(t)\, \dfrac{Z_1'(x)}{Z_0'(x)} - U_0(t)\, U_1(t).$

From (11.13) and (11.4),

$$|R_0'(x, t)| \leq \left[\frac{N' M^2}{m'} \; \frac{[1 + \nu\chi(x)]^{4L+2}}{[\nu\chi(x)]^{4L+2}} \; \frac{[\nu\chi(t)]^{4L+2}}{[1 + \nu\chi(t)]^{4L+3}} \right.$$

$$\left. \times \exp\{-2\nu[\chi(x) - \chi(t)]\} + \frac{MN}{1 + \nu\chi(t)} \right] \frac{\chi(t)}{\chi'(t)} \, ,$$

or

$$|R_0'(x, t)| \leq \frac{A(1+t)^{\frac{1}{2}}}{\nu} \qquad -1 < t \leq x \leq -\epsilon, \quad \nu > \nu_0$$

Hence, from (10.9), (11.8), and (11.16),

$$\left| \frac{z_0'}{Z_0'} - 1 \right| \leq \frac{AB}{\nu} \left(\int_{-1}^{x} (1+t)^{-\frac{1}{2}} dt \right) \left\{ 1 + \frac{2AB(1+x)^{\frac{1}{2}}/\nu}{1 - [2AB(1+x)^{\frac{1}{2}}/\nu]} \right\}$$

and it follows that

$$z_0'(x) = Z_0'(x) \left[1 + O\left(\frac{1}{\nu}\right) \right] \qquad -1 \leq x \leq -\epsilon, \quad \nu > \nu_0,$$

As in sec. 5,

(11.18) $u_0' = U_0' \left[1 + O\left(\dfrac{1}{\nu}\right) \right] + \dfrac{1}{2} \left(\ln \dfrac{\chi'}{\chi} \right)' U_0\, O\left(\dfrac{1}{\nu}\right)$

Now, $\ln(\chi'/\chi)$ is a monotone increasing function of x, $-1 < x < 0$. Furthermore, from (11.4) and (11.13), there exist positive constants m, m', M such that

$$\frac{U_0(x)}{U_0'(x)}[\ln(\chi'/\chi)]' = \frac{(\chi/\chi')^{\frac{1}{2}} I_{2L+1}(\nu\chi)\,[\ln(\chi'/\chi)]'}{(\chi/\chi')^{\frac{1}{2}} I_{2L+1}'(\nu\chi) + [(\chi/\chi')^{\frac{1}{2}}]' I_{2L+1}(\nu\chi)}$$

$$\leq \frac{\nu^{-1} M[\ln(\chi'/\chi)]'}{m'\chi'[1+1/\nu\chi] + \frac{1}{2}m\,\nu^{-1}[\ln(\chi/\chi')]'}$$

which is bounded, $-1 < x < 0$, $\nu > \nu_0$. Hence, (11.18) may be written

$$(11.19)\quad u_0'(x) = U_0'(x)\left[1 + O\left(\frac{1}{\nu}\right)\right] \qquad -1 < x \leq 0, \quad \nu > \nu_0$$

12. The solution $y_a(x)$

Suppose that a is a number in the interval $-1 < a < 0$. Let Y_0 and Y_{-1} be defined by (2.13), and let y_a be defined by the integral equation

$$(12.1)\quad y_a(x) = Y_0(x) + \int_a^x K(x, t)\, G(t)\, y_a(t)\, dt,$$

where $K(x, t)$ is given by (2.15) for $m = 1$, and $G(t)$ is given by (2.12). If $y_a(x)$ is an analytic solution of (12.1) for $a \leq x$, then $y_a(x)$ will satisfy the differential equation (9.1) and have the same behavior at $x = a$ as $Y_0(x)$.

The kernel $K(x, t)$ will now be estimated for the purpose of solving (12.1) by successive approximations. From (2.13) and (4.4),

$$(12.2)\quad |Y_0(x)| \leq \Phi(x) = \left|\frac{M(0)\exp(-\frac{2}{3}\nu[-\phi(x)]^{3/2})}{[\phi'(x)]^{1/2}\,[\frac{1}{4} + \nu^{2/3}|\phi(x)|]^{1/4}}\right|$$

$$|Y_{-1}(x)| \leq \left|\frac{M(-\frac{2}{3}\pi)\exp(\frac{2}{3}\nu[-\phi(x)]^{3/2})}{[\phi'(x)]^{1/2}\,[\frac{1}{4} + \nu^{2/3}|\phi(x)|]^{1/4}}\right| \qquad -1 \leq x.$$

Equations (3.3) and (10.7) show that

$$(12.3)\quad \frac{2}{3}[-\phi(x)]^{3/2} = \begin{cases} -i\nu f(x) & x > 0 \\ -\chi(x) + \tfrac{1}{2}\pi & a \le x \le 0 \end{cases}$$

where $f(x)$ and $\chi(x)$ are given by (3.1) and (10.2) respectively. Substitution of (12.2) and (2.16) into the kernel (2.15) gives the inequality

$$(12.4)\quad |K(x,t)| \le \frac{2\pi M(0) M(-\frac{2}{3}\pi)}{\nu^{2/3}}$$

$$\times \frac{E(x,t)}{[\phi'(x)]^{1/2}[\tfrac{1}{4}+\nu^{2/3}|\phi(x)|]^{1/4}[\phi'(t)]^{1/2}[\tfrac{1}{4}+\nu^{2/3}|\phi(t)|]^{1/4}}$$

where

$$E(x,t) = |\exp(\nu\{\tfrac{2}{3}[-\phi(t)]^{3/2} - \tfrac{2}{3}[-\phi(x)]^{3/2}\})$$

$$+ \exp(-\nu\{\tfrac{2}{3}[-\phi(t)]^{3/2} - \tfrac{2}{3}[-\phi(x)]^{3/2}\})| .$$

The behavior of $E(x, t)$ will now be examined in the different possible intervals for x and t.

(i) For $a \le t \le x \le 0$, by (12.3),

$$E(x,t) \le |\exp\{-\nu[\chi(t)-\chi(x)]\}| + |\exp\{\nu[\chi(t)-\chi(x)]\}|$$

$$\le 1 + \exp\{\nu[\chi(x)-\chi(t)]\}$$

$$\le 2\exp\{\nu[\chi(x)-\chi(t)]\}$$

since $\chi(t) - \chi(x) \le 0.$ for $t \le x$ by (10.2).

(ii) For $0 \le t \le x$, by (12.3),

$$E(x,t) \le |\exp\{-i\nu[f(t)-f(x)]\}| + |\exp\{i\nu[f(t)-f(x)]\}|$$

$$\le 2$$

(iii) For $a \leq t \leq 0 < x$,

$$E(x,t) \leq |\exp\{\nu[-\chi(t) + \pi/2 + if(x)]\}| + |\exp\{-\nu[-\chi(t)+\pi/2+if(x)]\}|$$

$$< \exp\{-\nu[\chi(t) - \pi/2]\} + 1$$

$$< 2\exp\{-\nu[\chi(t) - \pi/2]\}.$$

It follows from (12.3) that in each case

$$E(x, t) \leq 2|\exp(-\tfrac{2}{3}\nu[-\phi(x)]^{3/2})|\ |\exp(\tfrac{2}{3}\nu(-\phi(t)]^{3/2})|$$

and hence that (12.4) can be written

$$(12.5)\quad |K(x, t)| \leq \frac{4\pi M(-\frac{2}{3}\pi)}{\nu^{2/3}}\Phi(x)\left|\frac{\exp(\frac{2}{3}\nu[-\phi(t)]^{3/2})}{[\phi'(t)]^{1/2}[\frac{1}{4} + \nu^{2/3}|\phi(t)|]^{1/4}}\right|$$

where $\Phi(x)$ is given by (12.2). From (4.6), (12.2), and (12.5),

$$(12.6)\quad |K(x, t)\, Y_0(t)\, G(t)| \leq \frac{4\pi M(-\frac{2}{3}\pi) B}{\nu^{2/3}}$$

$$\times \Phi(x)\frac{1}{\phi'(t)[\frac{1}{4} + \nu^{2/3}|\phi(t)|]^{1/2}(1+t)^2},$$

$$-1 < a \leq t \leq x.$$

A formal solution of the integral equation (12.1) is defined by the series

$$(12.7)\quad y_a(x) = \sum_{n=0}^{\infty} y_a^{(n)}(x),$$

where

$$(12.8)\quad y_a^{(0)}(x) = Y_0(x)$$

$$y_a^{(n)}(x) = \int_a^x K(x,t)\, G(t)\, y_a^{(n-1)}(t)\, dt \qquad n \geq 1.$$

For the case $n = 1$, the bound (12.6) gives

$$(12.9)\quad |y_a^{(1)}(x)| \le \frac{4\pi M(-\frac{2}{3}\pi) B}{\nu} \; \Phi(x) \int_a^x \frac{dt}{|t|^{1/2}\,(1+t)^{3/2}}.$$

The integral on the right converges to a function of x which is bounded above by a constant D for $a \le x$. Hence,

$$|y_a^{(1)}(x)| \le \frac{CD}{\nu}\,\Phi(x),$$

where

$$C = 4\pi M(-\tfrac{2}{3}\pi) B.$$

Again,

$$|K(x, t)\, G(t)\, y_a^{(1)}(t)| \le \frac{CD}{\nu}\, |K(x, t)\, \Phi(t)\, G(t)|,$$

so that by (12.2) and (12.5),

$$|y_a^{(2)}(x)| \le \left(\frac{CD}{\nu}\right)^2 \Phi(x).$$

By mathematical induction on n it follows that

$$|y_a^{(n)}(x)| \le \left(\frac{CD}{\nu}\right)^n \Phi(x) \qquad n \ge 1, \quad -1 < a \le x,$$

where the constants C and D are independent of x, ν, and n. Consequently for $\nu > \nu_0 > CD$, the series (12.7) is uniformly and absolutely convergent to a regular function, $-1 < x < \infty$, which represents the unique continuous solution of the integral equation (12.1). Further, for $\nu > \nu_0$,

$$|y_a(x) - Y_0(x)| \le \Phi(x)\, \frac{CD/\nu}{1 - CD/\nu} \qquad -1 < x < \infty,$$

so that

$$(12.10)\quad y_a(x) = Y_0(x) + \Phi(x)\, O\left(\frac{1}{\nu}\right) \qquad \nu > \nu_0, \quad -1 < x < \infty.$$

For the case $a \leq t \leq x \leq 0$, the Airy function $Ai\,[-\nu^{2/3}\,\phi(x)]$ has no zeros, and hence (4.4) gives

$$|Ai\,[-\nu^{2/3}\,\phi(x)]| \geq \frac{m(0)\exp\{-\frac{2}{3}\nu[-\phi(x)]^{3/2}\}}{(\frac{1}{4}+\nu^{2/3}\,|\phi(x)|)^{1/4}}$$

uniformly for $-1 \leq x \leq 0$. It is then seen from (12.2), (12.3) and (2.13) that

$$\frac{\Phi(x)}{Y_0(x)} \leq \frac{M(0)\exp\{\nu[\chi(x)-\pi/2]\}}{[\phi'(x)]^{1/2}[\frac{1}{4}+\nu^{2/3}\,|\phi(x)|)^{1/4}} \Bigg/ \frac{m(0)\exp\{\nu[\chi(x)-\pi/2]\}}{[\phi'(x)]^{1/4}[\frac{1}{4}+\nu^{2/3}|\phi(x)|]^{1/4}}$$

$$= \frac{M(0)}{m(0)} \qquad -1 \leq x \leq 0, \quad \nu > \nu_0.$$

Hence, by (12.10)

$$(12.11) \quad y_a(x) = Y_0(x)\left[1 + O\left(\frac{1}{\nu}\right)\right] \qquad -1 < a \leq x \leq 0, \quad \nu > \nu_0.$$

For $x > 0$, this argument does not apply because the Airy function has a sequence of zeros in this interval, and is therefore not uniformly bounded away from zero. However, it is known [4; p. B48] that the zeros z_k of the function $Bi\,(z)$ interlace with the zeros z_i of $Ai\,(z)$, $i,k = 1, 2, \ldots$ and this fact will be used to obtain a modification of the result (12.11) which will apply for $0 \leq x$. From the properties of Airy functions [4; p. B17], there exist positive functions $m(\theta)$, $M(\theta)$, $N(\theta)$, and $n(\theta)$ such that [see section 4], with $\zeta = \frac{2}{3}\,z^{3/2}$, we have

$$(12.12) \quad m(\theta) \leq |(\tfrac{1}{4}+|z|)^{\frac{1}{4}}\,e^{\zeta}\,Ai(z)| \leq M(\theta)$$

$$n(\theta) \leq |(\tfrac{1}{4}+|z|)^{\frac{1}{4}}\,e^{-\zeta}\,Bi(z)| \leq N(\theta)$$

for $-\pi \leq \theta = \arg z \leq \pi$, provided that z is excluded from small circles $|z - z_i| < \delta$ and $|z - z_k| < \delta$, i, $k = 1, 2, \ldots$ in the first and second equation of (12.12) respectively. The small positive number δ may be chosen so that these circles do not intersect. For z excluded from $|z - z_i| < \delta$,

$$m(\theta) \leq |(\tfrac{1}{4}+|z|)^{\frac{1}{4}}\,e^{\zeta}(Ai^{\,2}(z) + Bi^{2}(z))^{\frac{1}{2}}|,$$

while for z excluded from $|z - z_k| < \delta$, and in particular for z in

$$|z - z_i| < \delta,$$

we have

$$n(\theta) \leq |(\tfrac{1}{4} + (z))^{1/4} \, e^{-\zeta} \, (Ai^2(z) + Bi^2(z))^{1/2}|.$$

Hence, if

$$s(\theta) = \text{minimum}\{m(\theta), n(\theta)\},$$

it follows for

$$z = -\tfrac{2}{3}\,\nu\,[-\phi(x)]^{3/2} \qquad 0 \leq x, \quad \arg z = \pi$$

that

$$(12.13) \quad s(\pi) \leq (1 + \nu^{2/3}|\phi(x)|)^{1/4}([Ai(-\nu^{2/3}\phi(x))]^2 + [Bi(-\nu^{2/3}\phi(x))]^2)^{1/2}$$

uniformly for $0 \leq x$. If $Y_2(x)$ is defined by

$$Y_2(x) = [\phi'(x)]^{-1/2}\,\mathrm{Bi}(-\nu^{2/3}\phi(x)),$$

then by (12.13),

$$[Y_0^2(x) + Y_2^2(x)]^{\frac{1}{2}} \geq \frac{s(\pi)}{[\phi'(x)]^{\frac{1}{2}}\,(\tfrac{1}{4} + \nu^{2/3}|\phi(x)|)^{1/4}}\,.$$

Hence, from (12.2),

$$\frac{\Phi(x)}{[Y_0^2(x) + Y_2^2(x)]^{\frac{1}{2}}} \leq \frac{M(0)}{s(\pi)} \qquad 0 \leq x$$

so that from (12.10),

$$(12.14) \quad y_a(x) = Y_0(x) + [Y_0^2(x) + Y_2^2(x)]^{\frac{1}{2}}\;O(1/\nu)$$

uniformly for $0 \leq x$, $\nu > \nu_0$. This is an appropriate modification of (12.11) valid in particular near the zeros of $Ai(-\nu^{2/3}\phi(x))$. Furthermore, by differentiation of the integral equation (12.1), it follows that

$$y_a'(x) = Y_0'(x) + \int_a^x \frac{\partial K_0(x, t)}{\partial x}\,G(t)\,y_a(t)\,dt,$$

since $K_0(x, x) \equiv 0$. The functions Y_j and Y_j' are bounded, $j = 0, -1$; $a \leq x < X$, and hence the kernel $\partial K_0(x, t)/\partial x$ is bounded by A/ν. The integral term is then of order $1/\nu$ for large ν, so that

$$(12.15)\quad y_a'(x) = Y_0'(x)\,[1 + O(1/\nu)] \qquad a \leq x \leq 0; \quad \nu > \nu_0.$$

13. Asymptotic approximations for the standard Coulomb wave functions

The regular solution $F_L(\eta, \rho)$ of (9.1) has the behavior (1.4) near $\rho = 0$, or $x = -1$, where the constant $C_L(\eta)$ is given by (1.5). The identification of the solutions $F_L(\eta, \rho)$ and $U_0(x)$ will now be made by comparing their behavior at the point $\rho = 0$, assuming that ν is fixed. Let

$$(13.1)\quad u_0(x) = D_L(\eta)\, F_L(\eta, \rho) + E_L(\eta)\, G_L(\eta, \rho).$$

From (1.4),

$$(13.2)\quad F_L(\eta, \rho) = \nu^{L+1}\, C_L(\eta)\,(1 + x)^{L+1}\,[1 + O(1 + x)] \qquad \rho \to 0,$$

and from (10.3), (10.8), and the behavior of the modified Bessel function near the origin [6],

$$(13.3)\ U_0(x) = \left(\frac{\chi}{\chi'}\right)^{1/2} I_{2L+1}(\nu\chi) = \frac{(\nu/2)^{2L+1}}{(2L+1)!}\,\frac{\chi^{2L+3/2}}{\chi'^{1/2}}[1 + O(\chi)]$$

$$= \frac{\nu^{2L+1}\, 2^{1/2}}{(2L+1)!}\,(1 + x)^{L+1}\,[1 + O(1 + x)] \qquad x \to -1.$$

On account of (11.12),

$$(13.4)\quad u_0(x) = \frac{\nu^{2L+1}\, 2^{1/2}}{(2L+1)!}\,(1 + x)^{L+1}\,[1 + O\{(1 + x)^{1/2}\}] \qquad x \to -1.$$

Since $G_L(\eta, \rho)$ is unbounded at $\rho = 0$, it follows from (1.5), (13.1), (13.2), and (13.4) that

$$(13.5)\quad E_L(\eta) = 0; \qquad [D_L(\eta)]^{-1} = 2^{L-1/2}\,\nu^{-L}\,e^{-\nu\pi/4}\,|\Gamma(L + 1 + \tfrac{1}{2}\,i\nu)|\,.$$

Substitution of (13.5) into (13.1) gives the result

$$(13.6)\quad F_L(\eta,\rho) = 2^{L-\frac{1}{2}}\,\nu^{-L}\,e^{-\nu\pi/4}\,|\Gamma(L+1+\tfrac{1}{2}i\nu)|\,(\chi/\chi')^{\frac{1}{2}}$$

$$\times\, I_{2L+1}(\nu\chi)\{1+O[\nu^{-1}(1+x)^{\frac{1}{2}}]\}$$

$$\text{as}\qquad \nu\to\infty,\qquad -1<x\leq-\epsilon,$$

the O term being uniform in the stipulated range of x. From the asymptotic form for the gamma function with large imaginary part [3; p. 47], equation (13.6) can be written in the following simpler, but weaker form:

$$(13.7)\quad F_L(\eta,\rho) = e^{-\nu\pi/2}\left(\frac{\pi\nu\chi}{2\chi'}\right)^{\frac{1}{2}} I_{2L+1}(\nu\chi)\,[1+O(\nu^{-1})],$$

$$\nu\to\infty,\quad -1<x\leq-\epsilon.$$

Finally, if x is bounded away from -1, and hence χ bounded away from 0, the asymptotic form of the modified Bessel function $I_{2L+1}(\nu\chi)$ [6; p. 203 (2)] leads to the result

$$(13.8)\quad F_L(\eta,\rho) = \tfrac{1}{2}\,[\chi'(x)]^{-\frac{1}{2}}\exp[\nu(\chi'(x)-\pi/2]\,[1+O(\nu^{-1})]$$

$$\nu\to\infty,\quad -1+\epsilon\leq x\leq-\epsilon<0.$$

Differentiation of (13.1) with respect to x and use of (11.19) gives

$$U_0'(x)\,[1+O(\nu^{-1})] + U_0(x)\,O(\nu^{-1}) = D_L\,F_L'(\eta,\rho).$$

From the discussion below (11.18), it follows that $U_0(x)/U_0'(x)$ is bounded for $a\leq x$, and hence the asymptotic forms (13.6), (13.7), and (13.8) may be formally differentiated with respect to x.

An asymptotic approximation for $F_L(\eta,\rho)$ in terms of the Airy function $Ai(-\nu^{2/3}\phi(x))$ will now be obtained by a comparison of (12.11) with the Bessel function approximation (13.8) for $-1<x<0$, and by a comparison of (11.14) with the approximation (6.15) for $x\geq 0$. Since the function $y_a(x)$ given in sec. 12 is a solution of (9.1), there exist constants C_1 and C_2 such that

$$(13.9)\quad y_a(x) = C_1\,F_L(\eta,\rho) + C_2\,G_L(\eta,\rho).$$

In this section, primes on all functions denote differentiation with respect to x,

$$f' = \frac{df}{dx} = \nu \frac{df}{d\rho}.$$

The abbreviation

$$[F_L]_a = [F_L(\eta, \rho)]_{x=a}$$

will be used for the regular and irregular Coulomb wave functions and their derivatives. From (13.9),

$$(13.10)\quad y_a'(x) = C_1 F_L'(\eta, \rho) + C_2 G_L'(\eta, \rho).$$

For $x = a$, the linear equations (13.9) and (13.10) possess a unique solution

$$(13.11)\quad C_1 = W^{-1}(a) \begin{vmatrix} y_a(a) & [G_L]_a \\ y_a'(a) & [G_L']_a \end{vmatrix}; \quad C_2 = W^{-1}(a) \begin{vmatrix} [F_L]_a & y_a(a) \\ [F_L']_a & y_a'(a) \end{vmatrix}$$

since the Wronskian [5]

$$(13.12)\quad W(a) = \begin{vmatrix} [F_L]_a & [G_L]_a \\ [F_L']_a & [G_L']_a \end{vmatrix} = -\nu$$

of the linearly independent solutions F_L and G_L of (9.1) does not vanish.

The constant C_1 will now be evaluated within a relative error of order ν^{-1} $(\nu \to \infty)$, and the term $C_2 G_L$ in (13.9) will be shown to be of order $\nu^{-1} y_a$ $(\nu \to \infty)$. The procedure consists of substituting the asymptotic forms for y_a, F_L, and G_L obtained in sections 12, 13, and 6, respectively, into equations (13.11). From (2.13), (12.11), and the asymptotic form for $Ai[-\nu^{2/3} \phi(x)]$ [4], it follows that

$$(13.13)\quad y_a(a)=\frac{1}{[\phi'(a)]^{1/2}}\; Ai(-\nu^{2/3}\,\phi(a)]$$

$$=\frac{1}{2\pi^{1/2}\,\nu^{1/6}\,[-\phi(a)(\phi'(a))^2]^{1/2}}\;\exp\{-\tfrac{2}{3}\nu[-\phi(a)]^{3/2}\}[1+O(\nu^{-1})]$$

$$=\frac{1}{2\nu^{1/6}\,[\pi\,\chi'(a)]^{1/2}}\;\exp[\nu(\chi(a)-\pi/2)]\,[1+O(\nu^{-1})]\,.$$

Since formal differentiation has been established in sec. (12), the following asymptotic form for the derivative is valid:

$$(13.14)\quad y_a'(a)=\tfrac{1}{2}\,\nu^{5/6}\,\pi^{-1/2}\,[\chi'(a)]^{1/2}\exp\{\nu[\chi(a)-\pi/2]\}\,[1+O(\nu^{-1})].$$

From (13.8)

$$(13.15)\quad [F_L]_a=\frac{1}{2[\chi'(a)]^{1/2}}\;\exp\{\nu[\chi(a)-\pi/2]\}\,[1+O(\nu^{-1})]$$

$$[F_L']_a=\tfrac{1}{2}\,\nu\,[\chi'(a)]^{1/2}\exp\{\nu[\chi(a)-\pi/2]\}\,[1+O(\nu^{-1})]\qquad \nu>\nu_0.$$

From (6.15),

$$(13.16)\quad G_L(\eta,\rho)=\pi^{1/2}\,\nu^{1/6}\,[\phi'(x)]^{-1/2}\,Bi(-\nu^{2/3}\,\phi(x))\,[1+O(\nu^{-1})],\qquad -1+\epsilon\le x,\quad \nu>\nu_0.$$

Use of the asymptotic form for $Bi(-\nu^{2/3}\,\phi(x))$ leads to the results

$$(13.17)\quad [G_L]_a=\frac{1}{[\chi'(a)]^{1/2}}\;\exp\{-\nu[\chi(a)-\pi/2]\}\,[1+O(\nu^{-1})]$$

$$[G_L']_a=-\nu[\chi'(a)]^{1/2}\exp\{-\nu[\chi(a)-\pi/2]\}\,[1+O(\nu^{-1})],\quad \nu>\nu_0.$$

From (12.12), it follows

$$\left|\frac{Bi(-\nu^{2/3}\,\phi(x))}{Ai(-\nu^{2/3}\,\phi(x))}\right|\le\frac{N(0)}{m(0)}\exp\{-2\nu(\chi(x)-\pi/2)\}\qquad -1\le x\le 0,$$

so that from (13.13) and (13.16)

$$(13.19)\quad \left|\frac{[G_L]_x}{y_a(x)}\right| \leq \pi^{1/2}\,\nu^{1/6}\,\frac{N(0)}{m(0)}\,\exp\{-2\nu[\chi(x)-\pi/2]\}\,[1+O(\nu^{-1})],$$

$$\nu > \nu_0.$$

From (13.12), (13.13), (13.14), (13.15), and (13.17) the constants in (13.11) are found to be

$$(13.20)\quad C_1 = \pi^{-1/2}\,\nu^{-1/6}\,[1+O(\nu^{-1})] \qquad \nu > \nu_0$$

$$(13.21)\quad C_2 = \nu^{5/6}\,\exp\{2\nu[\chi(a)-\pi/2]\}\,O(\nu^{-1}) \qquad \nu > \nu_0.$$

It follows from (13.19) and (13.21) that

$$(13.22)\quad \frac{C_2\,G_L(\eta,\rho)}{y_a(x)} = \exp\{2\nu[\chi(a)-\chi(x)]\}\,O(\nu^{-1}) = O(\nu^{-1}),$$

$$a \leq x \leq 0, \qquad \nu > \nu_0.$$

Hence (13.9) gives the asymptotic form

$$C_1\,F_L(\eta,\rho) = y_a(x)\,[1+O(\nu^{-1})],$$

or by virtue of (2.13), (12.11), and (13.20)

$$(13.23)\quad F_L(\eta,\rho) = \pi^{1/2}\,(2\eta)^{1/6}\,[\phi'(x)]^{-1/2}\,Ai(-\nu^{2/3}\,\phi(x))\,[1+\phi(\nu^{-1})],$$

uniformly for $-1+\epsilon \leq x \leq 0$, $\nu > \nu_0$.

For $0 < x$, the above treatment does not apply because of the presence of zeros of the function $Ai(-\nu^{2/3}\,\phi(x))$. In this case, the result (12.14) with modified error term is compared with the approximation (6.15) for $F_L(\eta,\rho)$:

$$\begin{aligned}(13.24)\quad \frac{1}{C_1}y_a - F_L &= \pi^{1/2}\nu^{1/6}[\phi']^{-1/2}\{Bi(-\nu^{2/3}\phi)\cos\lambda \\ &\quad + Ai(-\nu^{2/3}\phi)[1-\sin\lambda]\} \\ &\quad + [\phi']^{-1/2}\{Ai^2(-\nu^{2/3}\phi)+Bi^2(-\nu^{2/3}\phi)\}^{1/2}O(\nu^{-1})\end{aligned}$$

$$0 < x, \qquad \nu > \nu_0,$$

where

$$(13.25)\quad \sin \lambda = 1 + O(\nu^{-1}); \quad \cos \lambda = O(\nu^{-1}) \qquad \nu \to \infty.$$

Now, by (12.12), for $0 < x$,

$$(13.26)\quad |[\phi'(x)]^{-1/2}\, Bi(-\nu^{2/3}\phi)| < \left| \frac{N(\pi)\exp\{\tfrac{2}{3}\nu[-\phi(x)]^{3/2}\}}{(\tfrac{1}{4} + \nu^{2/3}|\phi(x)|)^{1/4}[\phi'(x)]^{1/2}} \right|$$

$$< \frac{N(\pi)}{\nu^{1/6}[\phi(x)\,\phi'^2(x)]^{1/4}},$$

and a similar estimate for $[\phi'(x)]^{1/2}\, Ai(-\nu^{2/3}\phi(x))$. By substitution of (13.25) and (13.26) into (13.24),

$$(13.27)\quad \frac{1}{C_1} y_a(x) - F_L(\eta, \rho) = [\phi'(x)]^{-1/2}\{[Ai(-\nu^{2/3}\phi(x))]^2 + [Bi(-\nu^{2/3}\phi(x))]^2\}^{1/2}\, O(\nu^{-1}),$$

$$0 < x, \quad \nu > \nu_0,$$

so that the result (13.23) with the error term modified as on the right of (13.27) is also valid for $0 < x$.

An asymptotic approximation for $G_L(\eta, \rho)$ in terms of the Airy function $Bi(-\nu^{2/3}\phi(x))$ is likewise obtained by comparison of (12.14) with (6.15) for all x in the interval $-1 + \epsilon \leq x$: a relation for $C_1^{-1}\, Bi(-\nu^{2/3}\phi(x)) - G_L(\eta, \rho)$ similar to (13.24) is written down, and results like (13.26) are used to estimate the Airy functions. A summary of the asymptotic formulas is given below:

$$(13.28)\quad F_L(\eta, \rho) = \pi^{1/2}(2\eta)^{1/6}[\phi'(x)]^{-1/2}\, Ai(-\nu^{2/3}\phi(x))[1 + O(\nu^{-1})]$$

$$-1 + \epsilon \leq x \leq 0, \quad \eta > \eta_0,$$

$$F_L(\eta, \rho) = \pi^{1/2}(2\eta)^{1/6}[\phi'(x)]^{-1/2}\, Ai(-\nu^{2/3}\phi(x)) + [\phi'(x)]^{-1/2}\{[Ai(-\nu^{2/3}\phi(x))]^2 + [Bi(-\nu^{2/3}\phi(x))]^2\}^{1/2} O(\nu^{-1})$$

$$0 \leq x, \quad \eta > \eta_0,$$

$$(13.28)\quad G_L(\eta,\rho)=\pi^{1/2}(2\eta)^{1/6}[\phi'(x)]^{-1/2}Bi(-\nu^{2/3}\phi(x))$$
$$+[\phi'(x)]^{-1/2}\{[Ai(-\nu^{2/3}\phi(x))]^2$$
$$+[Bi(-\nu^{2/3}\phi(x))]^2\}^{1/2}O(\nu^{-1})\quad -1+\epsilon\le x,\quad \eta>\eta_0,$$

where

$$x=\frac{\rho-2\eta}{2\eta}.$$

The asymptotic forms (13.28) are significant approximations to the Coulomb wave functions in the interval to the left of the transition point. Furthermore, they provide simplifications of the results (6.15) to the right of the transition point. However, the order term in (13.28) does not tend to zero as ρ tends to infinity for fixed η, and in this sense the result is weaker than (6.15). The two sets of formulas (6.15) and (13.28) then complement each other.

14. Numerical results

The dominant term in the asymptotic forms (13.28) for the regular and and irregular Coulomb wave functions are tabulated on the transition line $\rho = 2\eta$ for $L = 0$ and for values of η between 2 and 200. Some values of the functions are also given off the transition line for $L = 0$, $\eta = 3, 5$, $\rho = 1(1)10$.

For $L = 0$, $\rho = 2\eta$, (13.28) reduces to

$$F_0(\eta, 2\eta)\sim\pi^{1/2}(2\eta)^{1/6}Ai(0)$$
$$G_0(\eta, 2\eta)\sim\pi^{1/2}(2\eta)^{1/6}Bi(0).$$

Because of (2.9), equations (13.28) may be written

$$F_0(\eta,\rho)\sim\pi^{1/2}(2\eta)^{1/6}\left[\frac{1+x}{x}\phi(x)\right]^{1/4}Ai[-(2\eta)^{2/3}\phi(x)]$$

$$G_0(\eta,\rho)\sim\pi^{1/2}(2\eta)^{1/6}\left[\frac{1+x}{x}\phi(x)\right]^{1/4}Bi[-(2\eta)^{2/3}\phi(x)]$$

where $\phi(x)$ is given by (3.3) for $0\le x$, and is given in sec. 8 for $-1\le x\le 0$. Also, $\phi(x)$ is tabulated in Table II, sec. 8. The comparison values F_0^{TAB} and G_0^{TAB} are obtained as in sec. 8.

TABULATION OF ASYMPTOTIC APPROXIMATIONS OF COULOMB WAVE FUNCTIONS

$\rho = 2\eta = 4(2)26,\ 30(10)80,\ 100(20)200(40)400$

$F_0(\eta, 2\eta) \sim \pi^{1/2} (2\eta)^{1/6} Ai(0)$, $\quad G_0(\eta, 2\eta) = \pi^{1/2} (2\eta)^{1/6} Bi(0)$.

η	$\pi^{1/2}(2\eta)^{1/6}$	$F_0(\eta, 2\eta)$	F_0^{TAB}	$G_0(\eta, 2\eta)$	G_0^{TAB}
2	2.2331	0.7928	0.7752	1.3732	1.3975
3	2.3892	.8482	.8376	1.4692	1.4847
4	2.5066	.8899	.8825	1.5414	1.5526
5	2.6016	.9236	.9180	1.5998	1.6085
6	2.6819	.9522	.9476	1.6492	1.6562
7	2.7517	.9769	.9731	1.6921	1.6980
8	2.8136	.9989	.9957	1.7302	1.7353
9	2.8694	1.0187	1.0159	1.7645	1.7689
10	2.9203	1.0368	1.0343	1.7958	1.8000
11	2.9669	1.0533	1.0511	1.8244	1.8280
12	3.0103	1.0687	1.0667	1.8511	1.8543
13	3.0507	1.0831	1.0813	1.8760	1.8789
15	3.1243	1.1092	1.1077	1.9212	1.9238
20	3.2778	1.1637	1.1626	2.0156	2.0174
25	3.4020	1.2078	1.2070	2.0920	2.0934
30	3.5070	1.2451	1.244	2.1566	2.157
35	3.5983	1.2775	1.277	2.2127	2.213
40	3.6793	1.3063	1.306	2.2625	2.262
50	3.8186	1.3557	1.355	2.3482	2.348
60	3.9364	1.3975	1.397	2.4206	2.420
70	4.0389	1.4339	1.434	2.4836	2.483
80	4.1296	1.4661	1.466	2.5394	2.539
90	4.2115	1.4952	1.495	2.5898	2.589
100	4.2861	1.5217	1.522	2.6357	2.635
120	4.4184	1.5687	1.569	2.7170	2.716
140	4.5334	1.6095	1.609	2.7877	2.787
160	4.6355	1.6457	1.646	2.8505	2.849
180	4.7273	1.6783	1.678	2.9070	2.906
200	4.8111	1.7081	1.708	2.9585	2.957

$\pi^{1/2} = 1.77245$ $\quad Ai(0) = 0.35503$ $\quad Bi(0) = 0.61493$

TABLES OF VALUES OF $F_0(\eta, \rho)$, $G_0(\eta, \rho)$

$\eta = 3,\ 5; \qquad \rho = 1(1)10$

$\eta = 3$

ρ	x	$F_0(3,\rho)$	F_0^{TAB}	$G_0(3,\rho)$	$F_0^{TAB.}$
1	−0.833	.00334	.00287	83.49	76.56
2	−0.667	.0223	.0215	16.02	16.39
3	−0.500	.0874	.0844	5.895	6.019
4	−0.333	.236	.231	3.109	3.144
5	−0.167	.497	.489	2.064	2.079
6	0.000	.848	.838	1.469	1.485
7	0.167	1.188	1.180	0.896	0.913
8	0.333	1.355	1.353	0.209	0.228
9	0.500	1.188	1.198	− 0.517	− 0.501
10	0.667	0.641	.660	− 1.069	− 1.060

$\eta = 5$

ρ	x	$F_0(5,\rho)$	F_0^{TAB}	$G_0(5,\rho)$	G_0^{TAB}
1	−0.9	.0000217	.0000203	7700.	8085.
2	−0.8	.000298	.000286	841.9	869.7
3	−0.7	.00194	.001883	169.4	173.5
4	−0.6	.00846	.00827	48.50	49.43
5	−0.5	.0282	.0277	17.92	18.19
6	−0.4	.0765	.0754	8.180	8.272
7	−0.3	.1758	.1735	4.510	4.547
8	−0.2	.3487	.3450	2.937	2.952
9	−0.1	.6052	.6001	2.143	2.151
10	0	.9236	.9179	1.599	1.601

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